

$$\left| \overline{a_n \xleftarrow{f_n} \dots \xleftarrow{f_2} a_1 \xleftarrow{f_1} a_0} \right| p: a_0 \rightarrow a_n$$

Exercise 9. A *quiver* is a tuple $G = (V, E, d, c)$, where V and E are sets, and $d, c: E \Rightarrow V$ are functions. For $a, b \in V$, a *path* from a to b in G , is a list $p = (a_n, f_n, a_{n-1}, f_{n-1}, \dots, a_1, f_1, a_0)$, for some $n \geq 0$, with $a_i \in V$ and $f_i \in E$ for each $0 \leq i \leq n$, such that $a_0 = a$, $a_n = b$, and, for each $0 < i \leq n$, $d(f_i) = a_{i-1}$ and $c(f_i) = a_i$; we write $p: a \rightarrow b$. The *category of paths* on G , PG , has set of objects V and set of morphisms the paths in G .

a. Define an appropriate identity and composition on PG and show that PG is a category.

b. Let C be a category. A *morphism* from G to C is a pair of functions $F_0: V \rightarrow \text{ob } C$ and $F_1: E \rightarrow \text{mor } C$ such that, for all $f \in E$, $F_1 f$ has domain $F_0(d f)$ and codomain $F_0(c f)$. Prove that, for any morphism (F_0, F_1) from G to C , there is a unique functor $\bar{F}: PG \rightarrow C$ such that $\bar{F}a = F_0 a$ for all $a \in V$ and $\bar{F}(ce, e, de) = F_1 e$ for all $e \in E$.

Remark. The category PG is also called the free category on G .

$$E = \{f, g, h, k\} \quad V = \{a, b, c\}$$

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ & \searrow g & \\ & & c \\ & \swarrow h & \\ k & & \end{array} \quad \begin{array}{l} d(f) = a \\ c(f) = b \\ \text{etc.} \end{array}$$

$\circ, \text{id}$

a)

$$\begin{array}{l} p: a \rightarrow b \quad \text{paths in } G \\ q: b \rightarrow c \end{array} \quad \rightsquigarrow \quad q \circ p: a \rightarrow c$$

defined by concatenation:

$$\begin{array}{c} p \\ b \xleftarrow{f_n} \dots \xleftarrow{f_1} a \end{array}$$

$$(c, g_n, \dots, g_1, b, f_n, \dots, f_1, a)$$

$$\begin{array}{c} q \\ c \xleftarrow{g_n} \dots \xleftarrow{g_1} b \end{array}$$

Identity in PG on $a \in V$: (a)

Then PG is a category.

$$P: \text{Quiver} \rightleftarrows \text{Cat}: U$$

$$f: a \rightarrow b \rightsquigarrow F_1 f: F_0 a \rightarrow F_0 b$$

b.) For any quiver G , the morphism $G \rightarrow PG$, which is the identity on objects and sends an edge $e: a \rightarrow b$ to the "path" (b, e, a) in PG , is a universal arrow.

10 Adjunctions

Exercise 32. Let P and Q be posets, viewed as thin categories. Let $f: P \rightarrow Q$ be a functor, which we recall is the same thing as a monotone function.

a. Write down what it means in this case that a functor $g: Q \rightarrow P$ is right adjoint to f .

b. Prove that, if f has a right adjoint, then f preserves any supremum that exists in P .

c. Let $i: \mathbb{Z} \hookrightarrow \mathbb{Q}$ be the inclusion function. Does it have a right adjoint? Does it have a left adjoint?

d. Same questions for the inclusion function $\mathbb{Q} \hookrightarrow \mathbb{R}$.

$$r \in \mathbb{R} \mapsto \max \{q \in \mathbb{Q} \mid i(q) \leq r\}?$$

$\sqrt{2} \quad 1 \quad 1,4 \quad 1,41 \dots$

(a)

$$\text{Hom}(f p, q) \cong \text{Hom}(p, g q) \text{ for all } p \in P, q \in Q$$

$$g \text{ is a functor such that for all } p \in P, q \in Q, \boxed{f(p) \leq q \text{ iff } p \leq g(q)}$$

left $\swarrow$ right $\searrow$

b. f is a left adjoint $\Rightarrow$ f preserves colimits

Let $S \subseteq P$ and $s_0 = \sup S$. We show: $f(s_0) = \sup f[S]$.

$f(s_0)$ is an upper bound of $f[S]$ since f is monotone: if $s \in S$, $s \leq s_0 \Rightarrow f(s) \leq f(s_0)$.

Let $t \in Q$ be an upper bound of $f[S]$. Then

for all $s \in S$, we have $f(s) \leq t \xRightarrow{\text{adj.}} s \leq g(t)$. Since $s_0 = \sup S$, $s_0 \leq g(t)$.

$$\xRightarrow{\text{adj.}} f(s_0) \leq t.$$

c. $i: \mathbb{Z} \hookrightarrow \mathbb{Q}$ A right adjoint $g: \mathbb{Q} \rightarrow \mathbb{Z}$ should send $q \in \mathbb{Q}$ to $\boxed{\max \{p \in \mathbb{Z} \mid i(p) \leq q\}}$
left ceiling $= \lfloor q \rfloor$ floor

Adjunction

$G : D \rightarrow C$ s.t. for every $c \in \text{ob } C$, there exists $d \in \text{ob } D$, $\eta_c : c \rightarrow Gd$
such that, for any $d' \in \text{ob } D$, $f : c \rightarrow Gd'$, there is a unique
 $\bar{f} : d \rightarrow d'$ such that

$$\begin{array}{ccc} & Gd & \\ \eta_c \uparrow & \searrow G\bar{f} & \\ c & \xrightarrow{f} & Gd' \end{array}$$

commutes:

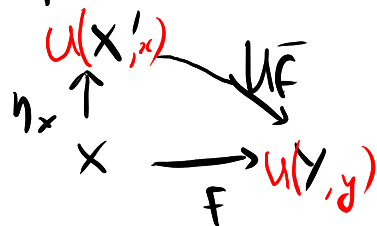
$$G\bar{f} \circ \eta_c = f$$

(F, G, θ) $F : C \rightleftharpoons D : G$ and $\theta_{c,d} : C(c, Gd) \xrightarrow{\sim} D(Fc, d)$ natural iso.
 C, D thin : $\underbrace{c \leq Gd \text{ iff } Fc \leq d}_{\text{for all } c \in C, d \in D}$

Exercise 33. Let $\mathbf{Set}_*$ be the category of *pointed sets*, whose objects are pairs (X, x) , with $x \in X$, and whose morphisms $(X, x) \rightarrow (Y, y)$ are functions $f: X \rightarrow Y$ such that $f(x) = y$. Prove that the functor $U: \mathbf{Set}_* \rightarrow \mathbf{Set}$, which sends (X, x) to X and f to f , has a left adjoint.

Option
Maybe

Need: for any set X , a pointed set (FX, x) and a universal arrow $X \xrightarrow{\eta_x} UFX$
 i.e. a function $\eta_x: X \rightarrow FX$ such that, for any pointed set (Y, y)
 and any $f: X \rightarrow Y$ there exists a unique $\bar{f}: (FX, x) \rightarrow (Y, y)$
 such that $U\bar{f} \circ \eta_x = f$ commutes:

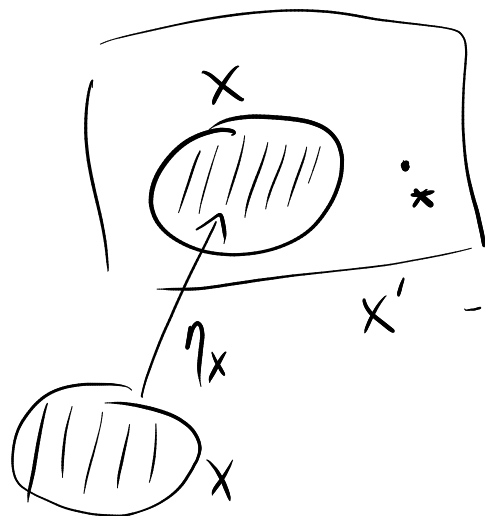


Add to X one new value: $X' := X + \{*\}$, $x := *$, $\eta_x: X \hookrightarrow X'$ the inclusion.

Let $f: X \rightarrow U(Y, y)$ be a function. Define $\bar{f}: (X', x) \rightarrow (Y, y)$ by:

$$\bar{f}(z) := \begin{cases} f(z) & \text{if } z \in X \\ y & \text{if } z = * \end{cases}$$

$$\bar{f}(z) = \begin{cases} f(z') & \text{if } z = \eta_x(z') \text{ for } z' \in X \\ y & \text{otherwise} \end{cases}$$



33 Then $U\bar{f} \circ \eta_x = f$ and $\bar{f}$ is the unique arrow in $\mathbf{Set}_*$ with this property.

$$\begin{array}{ccc}
 F : \text{Set} & \rightleftharpoons & \text{Set}_* : \mathcal{U} \\
 X & \mapsto & (X + \{*_X\}, *_X) \\
 f \downarrow & & \downarrow Ff \\
 Y & & (Y + \{*_Y\}, *_Y)
 \end{array}$$

$$\begin{array}{ccc}
 FX & \xrightarrow{\overline{f}} & FY \\
 \eta_X \uparrow & \nearrow f' & \uparrow \eta_Y \\
 UFX & \xrightarrow{Uf'} & UFY \\
 X & \xrightarrow{F} & Y
 \end{array}
 \quad \text{where } \overline{f} = Ff$$

$$\text{Set}_*(FX, (Y, y)) \cong \text{Set}(X, U(Y, y))$$

Exercise 34. Prove that the forgetful functor $U: \mathbf{Mon} \rightarrow \mathbf{Set}$ has a left adjoint.

List

let Σ be a set. We want: a function $\eta_\Sigma: \Sigma \rightarrow UF\Sigma$ for some monoid $F\Sigma$ such that $\forall f: \Sigma \rightarrow UM$, M a monoid, $\exists! \bar{f}: F\Sigma \rightarrow M$ in Mon such that $U\bar{f} \circ \eta_\Sigma = f$.

Left adjoints are "free constructions"

$$\begin{array}{ccc} UF\Sigma & & U\bar{f} \\ \eta_\Sigma \uparrow & \searrow & \downarrow \\ \Sigma & \xrightarrow{f} & UM \\ & \text{in Set} & \end{array} \quad \begin{array}{c} F\Sigma \\ \downarrow \bar{f} \\ M \\ \text{in Mon} \end{array}$$

$$\begin{array}{ccc} \mathbf{Vect}_R & \xrightarrow{U} & \mathbf{Set} \\ & \xleftarrow{\bar{F}} & \\ R^X & \xleftarrow{\quad} & X \text{ finite} \end{array}$$

$$F\Sigma := \Sigma^* \quad \left| \begin{array}{l} \eta_\Sigma: \Sigma \rightarrow U\Sigma^* \\ a \mapsto [a] \end{array} \right|$$

Let $f: \Sigma \rightarrow UM$ a function.

$\bar{f}: \Sigma^* \rightarrow M$ is the morphism defined by:

$$\begin{cases} w = [w_0, \dots, w_n] \mapsto \bar{f}(w) := f(w_0) \cdot \dots \cdot f(w_n) \\ w = [] \mapsto \bar{f}(w) := 1_M \end{cases}$$

Clearly, $U\bar{f} \circ \eta_\Sigma = f$, since $\bar{f}([a]) = f(a)$.

Also, if $g: \Sigma^* \rightarrow M$ is any morphism such that $g([a]) = f(a)$ for all $a \in \Sigma$, then $g = \bar{f}$. (By induction on the length of the input.)

The functor $F: \text{Set} \rightarrow \text{Mon}$ sends Σ to Σ^* and $f: \Sigma \rightarrow \Delta$ to $\bar{f}: \Sigma^* \rightarrow \Delta^*$,
 the unique morphism such that for each $a \in \Sigma$, $\bar{f}([a]) = [f(a)]$.

This means: $\bar{f}([w_0, \dots, w_n]) = [f(w_0), \dots, f(w_n)]$ "map f " : $\text{List } \Sigma \rightarrow \text{List } \Delta$.

We then also have: $\theta_{\Sigma, M}: \text{Mon}(F\Sigma, M) \cong \text{Set}(\Sigma, UM)$.

$$g: F\Sigma \rightarrow M \mapsto (Ug) \circ \eta_\Sigma$$

$$\begin{array}{ccc} & UF\Sigma & \\ \eta_\Sigma \uparrow & & \searrow Ug \\ \Sigma & \rightarrow & UM \end{array}$$

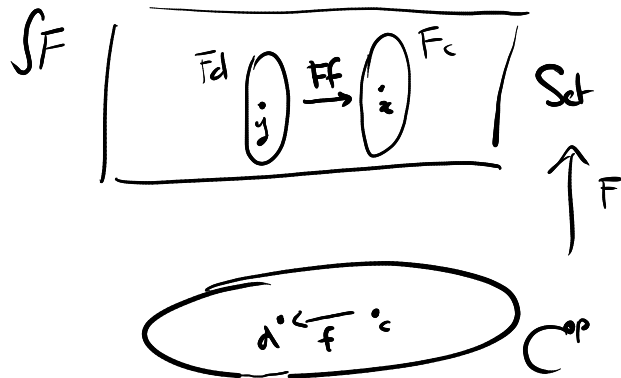
η_Σ is a universal arrow means that $\theta_{\Sigma, M}$ is a bijection for all M .

11 Universal elements and arrows

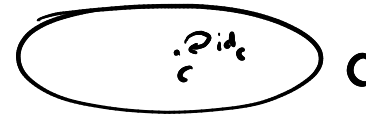
In this section, let C be a category and $F: C^{op} \rightarrow \mathbf{Set}$ a functor.

Definition G. An *element* of the functor F is a pair (c, x) , where c is an object of C and x is an element of the set Fc . Given two elements (c, x) and (d, y) of the functor F , a *morphism* from (c, x) to (d, y) is a morphism $f: c \rightarrow d$ in C such that $Ff(y) = x$. We write $\int F$ for the category of elements of the functor F .

Exercise 38. Prove that $\int F$ is a category.

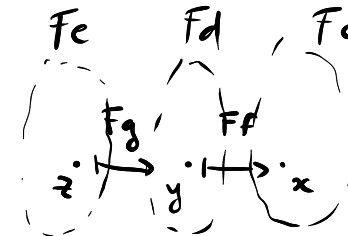


$$id_c: (c, x) \rightarrow (c, x)$$

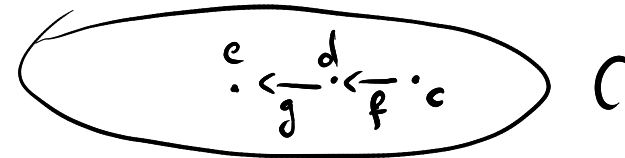


If $f: (c, x) \rightarrow (d, y)$ in $\int C$
and $g: (d, y) \rightarrow (e, z)$ in $\int C$
then $g \circ f: (c, x) \rightarrow (e, z)$.

This defines a category structure
on $\int C$.



$$\begin{aligned} F(g \circ f)(z) &= \\ (Ff \circ Fg)(z) &= \\ Ff(y) &= x. \end{aligned}$$



Definition H. An element (c, x) of F is *universal* if the unique (by the Yoneda Lemma) natural transformation, $\alpha: \mathbf{y}c \Rightarrow F$, such that $\mathbf{e}_{c,F}(\alpha) = x$, is an isomorphism.

HW **Exercise 39.** Let (c, x) be an element of F . Show that (c, x) is universal if, and only if, (c, x) is a terminal object of $\int F$.

for Oct 21