

Introduction to Categories

Lecture 6

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Adjunctions

Let $F \dashv G$ be an adjunction with $F: \mathbf{C} \rightarrow \mathbf{D}$ and $G: \mathbf{D} \rightarrow \mathbf{C}$. Then we have:

- a natural transformation $\eta: \text{Id} \Rightarrow GF: \mathbf{C} \rightarrow \mathbf{C}$, called *unit*;
- a natural transformation $\epsilon: FG \Rightarrow \text{Id}: \mathbf{D} \rightarrow \mathbf{D}$, called *counit*;
- a natural isomorphism $\Theta_{X,A}: \mathbf{C}(X, GA) \cong \mathbf{D}(FX, A)$.

This data satisfies the following properties:

- for every $g: X \rightarrow GA$, we have $\Theta(g) = \epsilon_X \circ Fg$;
- for every $f: FX \rightarrow A$, we have $\Theta^{-1}(f) = Gf \circ \eta_X$;
- for every $A \in \text{Ob}(\mathbf{D})$, we have $\Theta(\text{id}_{GA}) = \epsilon_A$;
- for every $X \in \text{Ob}(\mathbf{C})$, we have $\Theta^{-1}(\text{id}_{FX}) = \eta_X$;
- for every $X \in \text{Ob}(\mathbf{C})$, we have $\epsilon_{FX} \circ F\eta_X = \text{id}_{FX}$;
- for every $A \in \text{Ob}(\mathbf{D})$, we have $G\epsilon_A \circ \eta_{GA} = \text{id}_{GA}$;

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- for every $A \in \text{Ob}(\mathbf{D})$, we have $\Theta(\text{id}_{GA}) = \epsilon_A$;
- for every $X \in \text{Ob}(\mathbf{C})$, we have $\Theta^{-1}(\text{id}_{FX}) = \eta_X$;
- for every $X \in \text{Ob}(\mathbf{C})$, we have $\epsilon_{FX} \circ F\eta_X = \text{id}_{FX}$;
- for every $A \in \text{Ob}(\mathbf{D})$, we have $G\epsilon_A \circ \eta_{GA} = \text{id}_{GA}$;

Note: as mentioned previously, only a small part of this data already determines the rest. In other words, this is not a minimal presentation of an adjunction.

Adjunctions and (co)limits

A less trivial fact about adjunctions:

- **Proposition:** Left adjoints preserve colimits.
- **Proposition:** Right adjoints preserve limits.

Let's prove the first statement. The second one can be proven analogously.

Left adjoints preserve colimits

Proof. Let $F \dashv G$ be an adjunction with $F: \mathbf{C} \rightarrow \mathbf{D}$ and $G: \mathbf{D} \rightarrow \mathbf{C}$.

Let $D: \mathbf{I} \rightarrow \mathbf{C}$ be a diagram. We write $D_I \stackrel{\text{def}}{=} D(I)$ for the object mapping and for a morphism $i: I \rightarrow J$ in $\mathbf{I}$, we write $d_i \stackrel{\text{def}}{=} D(i): D_I \rightarrow D_J$ for the morphism mapping. Assume that the diagram D has a colimit in $\mathbf{C}$ and let

$$c \stackrel{\text{def}}{=} (C, \{c_I: D_I \rightarrow C \mid I \in \text{Ob}(\mathbf{I})\})$$

be its colimiting cocone. Since $F: \mathbf{C} \rightarrow \mathbf{D}$ is a functor, it is obvious that

$$Fc \stackrel{\text{def}}{=} (FC, \{Fc_I: FD_I \rightarrow FC \mid I \in \text{Ob}(\mathbf{I})\})$$

is a cocone for the diagram $FD: \mathbf{I} \rightarrow \mathbf{D}$. We have to show that it is the colimiting one.

Left adjoints preserve colimits (contd.)

Towards this end, assume that

$$z \stackrel{\text{def}}{=} (Z, \{z_I: FD_I \rightarrow Z \mid I \in \text{Ob}(\mathbf{I})\})$$

is a cocone for the diagram FD . We have to show that there is a unique cocone morphism $\hat{u}: Fc \rightarrow z$, i.e. a morphism $\hat{u}: FC \rightarrow Z$ in $\mathbf{D}$ such that $\hat{u} \circ Fc_I = z_I$ for every $I \in \text{Ob}(\mathbf{I})$. Since we have a natural isomorphism $\Theta_{X,A}: \mathbf{C}(X, GA) \cong \mathbf{D}(FX, A)$, we may look for a candidate morphism $u: C \rightarrow GZ$ in $\mathbf{C}$. We can build another cocone for D (in $\mathbf{C}$) by using Gz and the natural transformation $\eta: \text{Id} \rightarrow GF$. Let

$$z' \stackrel{\text{def}}{=} (GZ, \{Gz_I \circ \eta_{D_I}: D_I \rightarrow GZ \mid I \in \text{Ob}(\mathbf{I})\}).$$

We verify that z' is indeed a cocone for D :

$$\begin{aligned} Gz_J \circ \eta_{D_J} \circ d_i &= Gz_J \circ GFd_i \circ \eta_{D_I} && \text{(naturality)} \\ &= G(z_J \circ Fd_i) \circ \eta_{D_I} && \text{(functoriality)} \\ &= Gz_I \circ \eta_{D_I} && (z \text{ is a cocone of } FD) \end{aligned}$$

for any morphism $i: I \rightarrow J$ in $\mathbf{I}$, as required.

Left adjoints preserve colimits (contd.)

It follows that there is a *unique* cocone morphism $u: c \rightarrow z'$, i.e. a morphism $u: C \rightarrow GZ$ such that $Gz_I \circ \eta_{D_I} = u \circ c_I$ for every $I \in \text{Ob}(\mathbf{I})$. We claim that $\hat{u} \stackrel{\text{def}}{=} \Theta_{C,Z}(u): FC \rightarrow Z$ is the unique cocone morphism $Fc \rightarrow z$. We first show that it is indeed a cocone morphism. For any $i: I \rightarrow J$ in $\mathbf{I}$, we have:

$$\begin{aligned}
 \hat{u} \circ Fc_I &= \Theta_{C,Z}(u) \circ Fc_I && \text{(definition)} \\
 &= \epsilon_Z \circ Fu \circ Fc_I && \text{(slide "Adjunctions")} \\
 &= \epsilon_Z \circ F(u \circ c_I) && \text{(functoriality)} \\
 &= \epsilon_Z \circ F(Gz_I \circ \eta_{D_I}) && \text{(cocone morphism)} \\
 &= \epsilon_Z \circ FGz_I \circ F\eta_{D_I} && \text{(functoriality)} \\
 &= z_I \circ \epsilon_{FD_I} \circ F\eta_{D_I} && \text{(naturality)} \\
 &= z_I \circ \text{id}_{FD_I} && \text{(slide "Adjunctions")} \\
 &= z_I
 \end{aligned}$$

as required.

Left adjoints preserve colimits (contd.)

To show uniqueness, assume that $\tilde{u}: Fc \rightarrow z$ is a cocone morphism. Next, consider the morphism $\Theta_{C,Z}^{-1}(\tilde{u}): C \rightarrow GZ$ in $\mathbf{C}$. We verify that it is a cocone morphism $c \rightarrow z'$. Indeed, we have that:

$$\begin{aligned}
 \Theta_{C,Z}^{-1}(\tilde{u}) \circ c_I &= G\tilde{u} \circ \eta_C \circ c_I && \text{(slide "Adjunctions")} \\
 &= G\tilde{u} \circ GFc_I \circ \eta_{D_I} && \text{(naturality)} \\
 &= G(\tilde{u} \circ Fc_I) \circ \eta_{D_I} && \text{(functoriality)} \\
 &= Gz_I \circ \eta_{D_I} && \text{(cocone morphism)}
 \end{aligned}$$

as required. But $u: c \rightarrow z'$ is the unique cocone morphism (couniversality of the colimit), therefore $u = \Theta_{C,Z}^{-1}(\tilde{u})$. Since $\Theta_{C,Z}$ is a bijection, we have

$$\hat{u} = \Theta_{C,Z}(u) = \Theta_{C,Z}(\Theta_{C,Z}^{-1}(\tilde{u})) = \tilde{u}$$

which shows uniqueness. □

Equivalence of categories

Definition. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. An *equivalence of categories* is given by the following data:

- a functor $F: \mathbf{C} \rightarrow \mathbf{D}$;
- a functor $G: \mathbf{D} \rightarrow \mathbf{C}$;
- a natural isomorphism $\eta: \text{Id} \rightarrow GF: \mathbf{C} \rightarrow \mathbf{C}$;
- a natural isomorphism $\epsilon: FG \rightarrow \text{Id}: \mathbf{D} \rightarrow \mathbf{D}$.

Remark: An equivalence of categories preserves all *categorical* properties.

Full and faithful functors

Definition. Let $\mathbf{C}$ and $\mathbf{D}$ be categories and $F: \mathbf{C} \rightarrow \mathbf{D}$ be a functor. We say that:

- F is *faithful* if for any two morphisms $f, g: X \rightarrow Y$ in $\mathbf{C}$, we have $F(f) = F(g) \implies f = g$. That is, the function $\mathbf{C}(X, Y) \rightarrow \mathbf{D}(FX, FY) :: f \mapsto Ff$ is injective.
- F is *full* if for any morphism $h: FX \rightarrow FY$ in $\mathbf{D}$, there exists a morphism $f: X \rightarrow Y$ in $\mathbf{C}$ such that $h = Ff$. That is, the function $\mathbf{C}(X, Y) \rightarrow \mathbf{D}(FX, FY) :: f \mapsto Ff$ is surjective.
- F is *essentially surjective* if for every object $D \in \text{Ob}(\mathbf{D})$ there exists an object $C \in \text{Ob}(\mathbf{C})$ such that $D \cong FC$.

Equivalence of Categories

Proposition. (Axiom of choice). Two categories **C** and **D** are equivalent iff there exists a functor $F: \mathbf{C} \rightarrow \mathbf{D}$ which is full, faithful, and essentially surjective.

Proof. Let's prove LHS $\implies$ RHS. Assume we have the same data as in the definition of equivalence. We can show that both F and G are full, faithful and essentially surjective. To show that F is faithful, assume that $f, g: X \rightarrow Y$ are two morphisms. Assume that $Ff = Fg$. Then:

$$\begin{aligned}
 Ff = Fg &\implies GFf = GFg \\
 &\implies GFf \circ \eta_X = GFg \circ \eta_X \\
 &\implies \eta_Y \circ f = \eta_Y \circ g && \text{(naturality)} \\
 &\implies \eta_Y^{-1} \circ \eta_Y \circ f = \eta_Y^{-1} \circ \eta_Y \circ g \\
 &\implies f = g
 \end{aligned}$$

as required. Similarly, G is also faithful.

Equivalence of Categories (contd.)

Before we prove fullness, let's prove that for any object $X \in \text{Ob}(\mathbf{C})$, we have that

$$GF\eta_X = \eta_{GFX} \quad (1)$$

First, we draw a diagram. From naturality of η , it follows that

$$\eta_{GFX} \circ \eta_X = GF\eta_X \circ \eta_X$$

Since η is a natural isomorphism, we can compose with its inverse and therefore

$$\eta_{GFX} \circ \eta_X \circ \eta_X^{-1} = GF\eta_X \circ \eta_X \circ \eta_X^{-1}$$

and therefore $\eta_{GFX} = GF\eta_X$.

Equivalence of Categories (contd.)

To prove fullness, let $h: FX \rightarrow FY$ be a morphism (in **D**). Next, consider the morphism $f \stackrel{\text{def}}{=} \eta_Y^{-1} \circ Gh \circ \eta_X: X \rightarrow Y$ in **C**. We will show that $Ff = h$. Since the functor G is faithful, it suffices to prove that $GFf = Gh$. We have:

$$\begin{aligned}
 GFf &= GF\eta_Y^{-1} \circ GFGh \circ GF\eta_X && \text{(definition + functoriality)} \\
 &= \eta_{GFY}^{-1} \circ GFGh \circ \eta_{GFX} && \text{(equation (1))} \\
 &= \eta_{GFY}^{-1} \circ \eta_{GFY} \circ Gh && \text{(naturality)} \\
 &= Gh
 \end{aligned}$$

as required. Therefore F is full and we can prove similarly that G is full. Essential surjectivity is obvious for both functors. □

Equivalence and adjunctions

Proposition. Let (F, G, η, ϵ) be an equivalence of the categories $\mathbf{C}$ and $\mathbf{D}$ (as in the definition). Then $F \dashv G$ and $G \dashv F$.

Proof. Since F is full and faithful and $\epsilon: FG \Rightarrow \text{Id}$ is a natural isomorphism, then for every $X \in \text{Ob}(\mathbf{C}), B \in \text{Ob}(\mathbf{D})$, we have bijections

$$\mathbf{C}(X, GB) \cong \mathbf{D}(FX, FGB) \cong \mathbf{D}(FX, B)$$

given by

$$f \mapsto Ff \mapsto \epsilon_B \circ Ff.$$

Therefore $\Theta: \mathbf{C}(X, GB) \cong \mathbf{D}(FX, B) :: f \mapsto \epsilon_B \circ Ff$ is family of bijections. We show that Θ is natural by verifying:

$$\begin{aligned} \Theta(Gh \circ f \circ g) &= \epsilon \circ FGh \circ Ff \circ Fg && \text{(definition + functoriality)} \\ &= h \circ \epsilon \circ Ff \circ Fg && \text{(naturality)} \\ &= h \circ \Theta(f) \circ Fg && \text{(definition)} \end{aligned}$$

as required (for simplicity I omitted indices). Therefore $F \dashv G$ and similarly $G \dashv F$. $\square$

Equivalence and adjunctions

Proposition. Let $F \dashv G$ be an adjunction such that both $F: \mathbf{C} \rightarrow \mathbf{D}$ and $G: \mathbf{D} \rightarrow \mathbf{C}$ are full and faithful. Then $\mathbf{C}$ and $\mathbf{D}$ are categorically equivalent.

Proof. Let X and Y be objects of $\mathbf{C}$. Since F is a fully faithful left adjoint, we have

$$\mathbf{C}(X, Y) \cong \mathbf{D}(FX, FY) \cong \mathbf{C}(X, GFY)$$

given by

$$f \mapsto Ff \mapsto \Theta^{-1}(Ff) = GFf \circ \eta_X = \eta_Y \circ f$$

where we used naturality of η at the end. Therefore the function

$$(\eta_Y \circ -): \mathbf{C}(X, Y) \rightarrow \mathbf{C}(X, GFY)$$

is a bijection for every object $X \in \text{Ob}(\mathbf{C})$. By Exercise 3, it follows that η_Y is an isomorphism. Since Y is arbitrary, it follows that $\eta: \text{Id} \Rightarrow GF$ is a natural isomorphism. Similarly, $\epsilon: FG \Rightarrow \text{Id}$ is also a natural isomorphism. Therefore, $\mathbf{C}$ and $\mathbf{D}$ are equivalent.

Cartesian Closed Categories

Definition. We say that a category $\mathbf{C}$ is *Cartesian closed* if it has finite products (i.e. a terminal object and binary products) and for every object $Y \in \text{Ob}(\mathbf{C})$, the functor $(- \times Y): \mathbf{C} \rightarrow \mathbf{C}$ is a left adjoint. In this situation, we write $[Y \rightarrow -]$ for its right adjoint and we therefore have a *family* of adjunctions, one for each Y ,

$$(- \times Y) \dashv [Y \rightarrow -]: \mathbf{C} \rightarrow \mathbf{C}$$

and therefore isomorphisms

$$\mathbf{C}(X \times Y, Z) \cong \mathbf{C}(X, [Y \rightarrow Z])$$

which are natural in X and Z for every Y .

Remark. In fact, one can prove that these bijections are natural in all three components X, Y, Z .

Cartesian Closed Categories

(Non-)Examples:

- The category **Set** is Cartesian closed.
- The category **Pos** is Cartesian closed.
- The terminal category **1** is Cartesian closed.
- Is the empty $\emptyset$ category cartesian closed?
- The category **Vect** is not Cartesian closed.