

Exercises

Category Theory, M1 MPRI, ENS Paris-Saclay

First period, 2026-27

1 Preorders and monoids as categories

Exercise 1 (Preorders as thin categories). A category C is *thin* if for any objects a, b of C , there is at most one morphism from a to b . A *preorder* is a set P equipped with a binary relation $\leq$ which is reflexive and transitive.

Prove that a small thin category is essentially the same thing as a preordered set. More precisely, define mutually inverse assignments between the collection of small thin categories and the collection of preorders.

Exercise 2 (Monoids as one-object categories). Prove that a small category with one object is essentially the same thing as a monoid.

2 Isos, epis, monos

Definition A. Let $f: a \rightarrow b$ be a morphism in a category C . For any object c of C , we define a function $f_c^*: C(b, c) \rightarrow C(a, c)$ by $f_c^*(h) := hf$ for any $h: b \rightarrow c$. We also define a function $f_c^+: C(c, a) \rightarrow C(c, b)$ by $f_c^+(g) := fg$ for any $g: c \rightarrow a$.

Exercise 3 (Isomorphisms in a category). Let $f: a \rightarrow b$ be a morphism in a category C . Prove that the following four properties are equivalent:

- (i) f is an isomorphism, i.e., there exists $g: b \rightarrow a$ such that $fg = \text{id}_b$ and $gf = \text{id}_a$;
- (ii) f has a left inverse and a right inverse, i.e., there exist $p: b \rightarrow a$ and $s: b \rightarrow a$ such that $fs = \text{id}_b$ and $pf = \text{id}_a$;
- (iii) for any object c of C , the function f_c^* is a bijection;
- (iv) for any object c of C , the function f_c^+ is a bijection.

Exercise 4 ($\text{Iso} \subsetneq \text{Mono} \cap \text{Epi}$). This exercise shows that the class of isomorphisms is included in the class of morphisms that are both mono and epi, and that this inclusion is strict in general.

- a. Prove that an isomorphism is both a monomorphism and an epimorphism.
- b. Prove that, in the category **Mon** of monoids and homomorphisms, the inclusion function $i: \mathbb{N} \hookrightarrow \mathbb{Z}$ is both a monomorphism and an epimorphism, but not an isomorphism.

Hint. Recall that a monoid homomorphism must preserve any existing inverses.

Exercise 5 (Split epimorphisms). Let C be a category. A morphism $p: a \rightarrow b$ in C is called a *split epimorphism* if it has a right inverse, i.e., if there exists $s: b \rightarrow a$ such that $ps = \text{id}_b$.

- Prove that a split epimorphism is an epimorphism.
- Give an example of a category C in which not every epimorphism is a split epimorphism.
- Prove that p is a split epimorphism if, and only if, for any object c of C , the function p_c^+ (see Definition A) is surjective.
- Show, without using the Axiom of Choice, that a function is surjective if, and only if, it is an epimorphism in the category **Set**.
- Recall that the Axiom of Choice (AC) states that, for any set $\mathcal{A}$ of non-empty sets, there exists a function f with domain $\mathcal{A}$ and such that, for every $A \in \mathcal{A}$, $f(A) \in A$. Prove that AC is equivalent to: in the category **Set**, every epimorphism is a split epimorphism.

3 Functors

Exercise 6. Let C be the category whose objects are the vector spaces $\mathbb{R}^d$ for each $d \in \mathbb{N}$, and whose morphisms are the linear transformations. Let $\mathbf{Mat}_{\mathbb{R}}$ be the category whose objects are natural numbers, whose morphisms from k to n are the $n \times k$ real matrices, and whose composition is given by matrix multiplication.

Define a functor F from $\mathbf{Mat}_{\mathbb{R}}$ to C which is an *isomorphism*, i.e., F is a bijection on objects and, for any objects n, k of $\mathbf{Mat}_{\mathbb{R}}$, the function $F: \mathbf{Mat}_{\mathbb{R}}(k, n) \rightarrow C(Fk, Fn)$ is a bijection.

Exercise 7. Let Σ be a set. Using Exercise 2, regard the monoid Σ^* as a one-object category. Denote by $\mathbf{Set}_f$ the category of finite sets and functions between them. Prove that the collection of functors $F: \Sigma^* \rightarrow \mathbf{Set}_f$ is in bijection with the collection of pairs (Q, δ) , where Q is a finite set and $\delta: \Sigma \times Q \rightarrow Q$ is a function.

Exercise 8. Let $F: C \rightarrow D$ be a functor.

- Prove that F preserves isomorphisms, i.e., if f is an iso in C , then Ff is an iso in D .
- Show that F does not necessarily preserve monomorphisms or epimorphisms.

Exercise 9. A *quiver* is a tuple $G = (V, E, d, c)$, where V and E are sets, and $d, c: E \rightrightarrows V$ are functions. For $a, b \in V$, a *path* from a to b in G , is a list $p = (a_n, f_n, a_{n-1}, f_{n-1}, \dots, a_1, f_1, a_0)$, for some $n \geq 0$, with $a_i \in V$ and $f_i \in E$ for each $0 \leq i \leq n$, such that $a_0 = a$, $a_n = b$, and, for each $0 < i \leq n$, $d(f_i) = a_{i-1}$ and $c(f_i) = a_i$; we write $p: a \rightarrow b$. The *category of paths* on G , PG , has set of objects V and set of morphisms the paths in G .

- Define an appropriate identity and composition on PG and show that PG is a category.
- Let C be a category. A *morphism* from G to C is a pair of functions $F_0: V \rightarrow \text{ob } C$ and $F_1: E \rightarrow \text{mor } C$ such that, for all $f \in E$, $F_1 f$ has domain $F_0(df)$ and codomain $F_0(cf)$. Prove that, for any morphism (F_0, F_1) from G to C , there is a unique functor $\bar{F}: PG \rightarrow C$ such that $\bar{F}a = F_0a$ for all $a \in V$ and $\bar{F}(ce, e, de) = F_1e$ for all $e \in E$.

Remark. The category PG is also called the *free category* on G .

4 Binary products and terminal objects

Exercise 10. Let P be a preorder, seen as a category (see Exercise 1).

- Let $t \in P$. Show that t is terminal if, and only if, t is maximal.
- Let $a, b, c \in P$. Show that c is a product of a and b if, and only if, c is a greatest lower bound (infimum) of a and b .

Exercise 11. a. Give an example of sets A and B such that the projection $A \times B \rightarrow B$ is not surjective.

- Conclude that product projections need not be epimorphisms.
- Show that the unique map to a terminal object need not be an epimorphism.

Definition B. Let I be an index set, C a category, a an object, and $(f_i: a \rightarrow b_i)_{i \in I}$ a family of morphisms with domain a . We say that the family $(f_i)_{i \in I}$ is *jointly monomorphic* if, for any pair of morphisms $g, h: x \rightrightarrows a$, if $f_i \circ g = f_i \circ h$ for every $i \in I$, then $g = h$.

We say that a jointly monomorphic family $(f_i: a \rightarrow b_i)_{i \in I}$ is *extremal* if, for any family of morphisms $(f'_i: a' \rightarrow b_i)_{i \in I}$ and any epimorphism $e: a \rightarrow a'$, if $f_i = f'_i \circ e$ for every $i \in I$, then e is an isomorphism.

Exercise 12. Let C be a category, let b_1, b_2 be objects in C and let $p_1: b_1 \times b_2 \rightarrow b_1$, $p_2: b_1 \times b_2 \rightarrow b_2$ be a product.

- Prove that the family (p_1, p_2) is jointly monomorphic and extremal.
- Let $(f_i: a \rightarrow b_i)_{i=1,2}$ be a family of morphisms with domain a . Prove that the family is jointly monomorphic if, and only if, $\langle f_1, f_2 \rangle$ is a monomorphism.

5 Binary coproducts and initial objects

Definition C. Let C be a category and a_1, a_2 objects of C . A pair of morphisms $i_1: a_1 \rightarrow b$ and $i_2: a_2 \rightarrow b$ is a *coproduct* if it is a product in C^{op} . An object a of C is *initial* if it is terminal in C^{op} .

Exercise 13. a. Write down a more concrete definition of ‘coproduct’ and of ‘initial’.

- Explain why coproducts and initial objects are unique up to isomorphism.
- Show that coproducts and an initial object exist in **Set**.
- What are a coproduct and initial object in a preorder?

Notation. A coproduct of two objects a_1, a_2 is denoted $a_1 + a_2$, and an initial object is denoted 0 . The copairing of two morphisms $f_1: a_1 \rightarrow c$ and $f_2: a_2 \rightarrow c$ is denoted $[f_1, f_2]: a_1 + a_2 \rightarrow c$. The functions $i_k: a_k \rightarrow a_1 + a_2$ are called *coprojections*, or also *injections* or *inclusions* (note that they are not necessarily monomorphisms, by the dual of Exercise 11).

Exercise 14. Let C be a category in which any two objects have a product and a coproduct.

- a. Show that, for any objects a, b, c , there exists a morphism $(a \times b) + (a \times c) \rightarrow a \times (b + c)$.
- b. Does there always exist an isomorphism?

Exercise 15. Consider the monoids $(\mathbb{N}, 0, +)$ and $(\{a, b\}^*, \epsilon, \cdot)$.

- a. Prove that $\{a, b\}^*$ is a coproduct of $\mathbb{N}$ and $\mathbb{N}$ in the category **Mon**.
- b. What is the coproduct of $\{a, b\}^*$ and $\mathbb{N}$ in **Mon**?
- c. What is the coproduct of $\mathbb{Z}$ and $\mathbb{Z}$ in the category of groups?

Exercise 16. Let Σ be a finite set. Consider the following category, which we denote $\Sigma/\mathbf{Mon}$:

- objects are pairs (M, f) , where M is a monoid, and $f: \Sigma \rightarrow M$ is a function from Σ to (the underlying set of) M ;
- a morphism $(M, f) \rightarrow (N, g)$ is a monoid homomorphism $h: M \rightarrow N$ such that $h \circ f = g$.

Show that $\Sigma/\mathbf{Mon}$ has an initial object.

Exercise 17. Let C be a category with a terminal object 1 and an initial object 0 . Prove that any morphism $f: 1 \rightarrow 0$ is an isomorphism.

6 Equalizers and coequalizers

Exercise 18. Prove that any equalizer is a monomorphism.

Exercise 19. Let $f, g: A \rightrightarrows B$ be two functions in **Set**. Define $\sim$ to be the smallest equivalence relation on B such that $f(a) \sim g(a)$ for every $a \in A$. Prove that the quotient map $e: B \rightarrow B/\sim$ is a coequalizer of f and g .

7 Pullbacks and pushouts

Definition D. The *walking span*, Λ , is the (poset) category with three objects and two non-identity morphisms as depicted on the left below. The *walking cospan*, V , is its opposite category, as depicted on the right below.



A *span* in a category C is a diagram of shape Λ , and a *cospan* is a diagram of shape V .

A *pushout*¹ is a colimit of a span, and a *pullback*² is a limit of a cospan.

Exercise 20. a. Give more explicit definitions of pushout and pullback.

¹In French: *somme amalgamée*.

²In French: *produit fibré*.

- b. Let $f_1: b_1 \rightarrow a$ and $f_2: b_2 \rightarrow a$ be morphisms in a category C . Suppose that a product $b_1 \times b_2$ of b_1 and b_2 exists in C , with projections $p_i: b_1 \times b_2 \rightarrow b_i$ for $i = 1, 2$, and that the pair of morphisms $f_1 p_1, f_2 p_2: b_1 \times b_2 \rightrightarrows a$ has an equalizer, $e: c \rightarrow b_1 \times b_2$. Prove that $(p_1 e, p_2 e)$ is a pullback of (f_1, f_2) .
- c. Let $f_1: B_1 \rightarrow A$ and $f_2: B_2 \rightarrow A$ be a cospan in **Set**. Show that there exists a pullback with summit $P := \{(b_1, b_2) \in B_1 \times B_2 \mid f_1(b_1) = f_2(b_2)\}$.
- d. Let $f_1: A \rightarrow B_1$ and $f_2: A \rightarrow B_2$ be a span in **Set**. Compute its pushout.

Exercise 21. Let C be a category containing the following two commuting squares.

$$\begin{array}{ccccc} a & \xrightarrow{p} & b & \xrightarrow{t} & e \\ \downarrow q & & \downarrow r & & \downarrow v \\ c & \xrightarrow{s} & d & \xrightarrow{u} & f \end{array}$$

Suppose that (r, t) is a pullback of (u, v) . Prove that (tp, q) is a pullback of (v, us) if, and only if, (p, q) is a pullback of (r, s) .

Exercise 22. Let C be a category in which pullbacks and a terminal object exist.

- Prove that binary products exist in C .
- Prove that all equalizers exist in C .

Exercise 23. Prove that a morphism $f: a \rightarrow b$ is mono if, and only if, the square

$$\begin{array}{ccc} a & \xrightarrow{\text{id}_a} & a \\ \downarrow \text{id}_a & & \downarrow f \\ a & \xrightarrow{f} & b \end{array}$$

is a pullback.

8 Limits in general

Exercise 24. The aim of this exercise is to prove that *a category with small products and equalizers is complete*.

Let C be a category and let $D: I \rightarrow C$ be a diagram in C , with I small. For each morphism t in I , write dt for its domain and ct for its codomain.

Suppose that the family of objects $(D_i)_{i \in \text{ob } I}$ has a product $(p_i: P \rightarrow D_i)_{i \in \text{ob } I}$ in C , and that the family of objects $(D_{ct})_{t \in \text{mor } I}$ has a product $(q_t: Q \rightarrow D_{ct})_{t \in \text{mor } I}$ in C .

Consider the morphisms $f, g: P \rightrightarrows Q$ defined as

$$f := \langle p_{ct} \rangle_{t \in \text{mor } I} \text{ and } g := \langle Dt \circ p_{dt} \rangle_{t \in \text{mor } I},$$

that is, $q_t \circ f = p_{ct}$ and $q_t \circ g = Dt \circ p_{dt}$ for every $t \in \text{mor } I$.

Suppose that $e: E \rightarrow P$ is an equalizer of the pair (f, g) .

Prove that $(p_i \circ e: E \rightarrow D_i)_{i \in \text{ob } I}$ is a limit cone for D .

Definition E. A poset $(I, \leq)$ is *down-directed* if I is non-empty and, for any $a, b \in I$, there exists $c \in I$ such that $c \leq a$ and $c \leq b$. A diagram $D: I \rightarrow C$ is *projective* if I is a down-directed poset. A category C has *projective limits* if any projective diagram in C has a limit.

Exercise 25. The aim of this exercise is to prove that *a category with finite products and projective limits has all small products*. Combined with the previous exercise, it follows that *a category with finite and projective limits is complete*.

Let $(a_j)_{j \in J}$ be a family of objects in a category C . Define I to be the set of finite subsets of J , ordered by reverse inclusion, i.e., for $S, T \in I$, $S \leq T$ if, and only if $T \subseteq S$.

- a. Prove that $(I, \leq)$ is down-directed.

For each S in I , suppose that $(p_i^S: D_S \rightarrow a_i)_{i \in S}$ is a product of $(a_i)_{i \in S}$ in C .

- b. Show that whenever $S, T \in I$ and $S \leq T$, there exists a unique morphism $D_{ST}: D_S \rightarrow D_T$ in C such that, for any $i \in T$, we have $p_i^T \circ D_{ST} = p_i^S$.
- c. Conclude that $D: I \rightarrow C$ is a diagram.

Suppose that $(q_S: P \rightarrow D_S)_{S \in I}$ is a limit of D .

- d. Show that $(p_j^{\{j\}} \circ q_{\{j\}}: P \rightarrow a_j)_{j \in J}$ is a product of the family $(a_j)_{j \in J}$.

Exercise 26. Let C be a small category in which all small families have products. Prove that C is thin and complete.

9 The Yoneda lemma

The exercises in this section lead to a proof of the Yoneda Lemma. We first examine two special cases: the case of a (totally ordered) thin category in Exercise 28 and the case of a monoid in Exercise 29. These exercises could be solved quickly by appealing to the result of Exercise 30.a, but it is instructive to also find direct proofs.

Recall from Definition A that, for a morphism $f: a \rightarrow b$ in a category C and c an object of C , we write $f_c^*: C(b, c) \rightarrow C(a, c)$ for the function defined by $f_c^*(h) := hf$ for any $h: b \rightarrow c$, and $f_c^+: C(c, a) \rightarrow C(c, b)$ for the function defined by $f_c^+(g) := fg$ for any $g: c \rightarrow a$.

Definition F. Let C be a (locally small, as we always assume by default) category.

For any object c of C , we write $\mathbf{y}c$ for the functor $C^{\text{op}} \rightarrow \mathbf{Set}$ which sends an object a of C to the set $(\mathbf{y}c)a := C(a, c)$, and a morphism $f: a \rightarrow b$ of C to the function $(\mathbf{y}c)f := f_c^*: C(b, c) \rightarrow C(a, c)$.

For any morphism $g: c \rightarrow c'$ of C , we write $\mathbf{y}g$ for the natural transformation $\mathbf{y}c \Rightarrow \mathbf{y}c'$ whose component at an object a of C is the function $(\mathbf{y}g)_a := g_a^+: C(a, c) \rightarrow C(a, c')$.

In what follows, when C is a category, we write $\widehat{C}$ for the category whose objects are functors $C^{\text{op}} \rightarrow \mathbf{Set}$, and whose morphisms are natural transformations between such functors.

Exercise 27. Let C be a category.

- a. Prove that, for any object c of C , $\mathbf{y}c$ is a functor.
- b. Prove that, for any morphism g of C , $\mathbf{y}g$ is a natural transformation.
- c. Prove that $\mathbf{y}: C \rightarrow \widehat{C}$ is a functor.

Exercise 28 (Yoneda lemma for ω). Let ω be the total order of natural numbers, viewed as a thin category.

- a. Show that functors $F: \omega^{\text{op}} \rightarrow \mathbf{Set}$ correspond bijectively to inverse sequences of sets

$$\cdots \longrightarrow X_{n+1} \xrightarrow{f_n} X_n \longrightarrow \cdots \longrightarrow X_2 \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0$$

i.e., the data of, for each $n \in \omega$, a set X_n , and a function $f_n: X_{n+1} \rightarrow X_n$.

- b. Describe, for any $n \in \omega$, the sequence corresponding to the functor $\mathbf{y}n: \omega^{\text{op}} \rightarrow \mathbf{Set}$.
- c. Let $n \in \omega$ and let $(f_n: X_{n+1} \rightarrow X_n)_{n \in \omega}$ be an inverse sequence of sets with corresponding functor $F: \omega^{\text{op}} \rightarrow \mathbf{Set}$. Prove that natural transformations $\mathbf{y}n \Rightarrow F$ are in bijective correspondence with elements of X_n .

Exercise 29 (Yoneda lemma for monoids). Let M be a monoid, viewed as a category with one object, denoted $\star$.

- a. Show that functors $F: M^{\text{op}} \rightarrow \mathbf{Set}$ correspond bijectively to *actions* of M on a set X . Here, by an action of M on X we mean a function $\bullet: X \times M \rightarrow X$ such that, for any $x \in X$, $x \bullet 1_M = x$, and for any $m, n \in M$, $x \bullet (mn) = (x \bullet m) \bullet n$.
- b. Describe the action corresponding to the functor $\mathbf{y}\star$.
- c. Let $F: M^{\text{op}} \rightarrow \mathbf{Set}$ be a functor and let $\bullet: X \times M \rightarrow X$ be the corresponding action. Give a bijection between X and the set of natural transformations $\mathbf{y}\star \Rightarrow F$.

Exercise 30 (The Yoneda Lemma in general). Let C be a category. For any functor $F: C^{\text{op}} \rightarrow \mathbf{Set}$ and any object c of C , we define a function

$$\begin{aligned} \mathbf{e}_{c,F}: \widehat{C}(\mathbf{y}c, F) &\rightarrow Fc \\ \alpha &\mapsto \alpha_c(\text{id}_c) . \end{aligned}$$

- a. Prove that $\mathbf{e}_{c,F}$ is a bijection by constructing an inverse function.
- b. Prove that $\mathbf{e}$ is natural in F , i.e., for any morphism $\beta: F \Rightarrow G$ in $\widehat{C}$, the diagram

$$\begin{array}{ccc} \widehat{C}(\mathbf{y}c, F) & \xrightarrow{\mathbf{e}_{c,F}} & Fc \\ \downarrow \beta_{\mathbf{y}c}^+ & & \downarrow \beta_c \\ \widehat{C}(\mathbf{y}c, G) & \xrightarrow{\mathbf{e}_{c,G}} & Gc \end{array}$$

commutes.

- c. Prove that $\mathbf{e}$ is natural in c , i.e., for any morphism $g: c \rightarrow d$ in C , the diagram

$$\begin{array}{ccc} \widehat{C}(\mathbf{y}d, F) & \xrightarrow{\mathbf{e}_{d,F}} & Fd \\ \downarrow (\mathbf{y}g)_F^* & & \downarrow Fg \\ \widehat{C}(\mathbf{y}c, F) & \xrightarrow{\mathbf{e}_{c,F}} & Fc \end{array}$$

commutes.

Exercise 31. Prove that the functor $\mathbf{y}: C \rightarrow \widehat{C}$ is full and faithful.

10 Adjunctions

Exercise 32. Let P and Q be posets, viewed as thin categories. Let $f: P \rightarrow Q$ be a functor, which we recall is the same thing as a monotone function.

- Write down what it means in this case that a functor $g: Q \rightarrow P$ is right adjoint to f .
- Prove that, if f has a right adjoint, then f preserves any supremum that exists in P .
- Let $i: \mathbb{Z} \hookrightarrow \mathbb{Q}$ be the inclusion function. Does it have a right adjoint? Does it have a left adjoint?
- Same questions for the inclusion function $\mathbb{Q} \hookrightarrow \mathbb{R}$.

Exercise 33. Let $\mathbf{Set}_*$ be the category of *pointed sets*, whose objects are pairs (X, x) , with $x \in X$, and whose morphisms $(X, x) \rightarrow (Y, y)$ are functions $f: X \rightarrow Y$ such that $f(x) = y$. Prove that the functor $U: \mathbf{Set}_* \rightarrow \mathbf{Set}$, which sends (X, x) to X and f to f , has a left adjoint.

Exercise 34. Prove that the forgetful functor $U: \mathbf{Mon} \rightarrow \mathbf{Set}$ has a left adjoint.

Exercise 35. Recall from Exercise 9 the definition of a category PG for any quiver G . (If you did not yet do Exercise 9, now is a good time.) A *morphism* from a quiver $G = (V_G, E_G, d_G, c_G)$ to a quiver $H = (V_H, E_H, d_H, c_H)$ is a pair of functions $f_0: V_G \rightarrow V_H$ and $f_1: E_G \rightarrow E_H$ such that, $d_H \circ f_1 = f_0 \circ d_G$ and $c_H \circ f_1 = f_0 \circ c_G$. This gives a category **Quiver** of quivers. Show that the forgetful functor $U: \mathbf{Cat} \rightarrow \mathbf{Quiver}$ from the category of small categories to the category of quivers has a left adjoint, which is given on objects by the construction $G \mapsto PG$.

Exercise 36. Let $F: C \rightleftarrows D: G$ be a pair of functors and, for each c in C and $d \in D$, let $\alpha_{c,d}: D(Fc, d) \xrightarrow{\cong} C(c, Gd)$ be a bijection. Prove that the following are equivalent:

- α is natural in both coordinates;
- for any morphisms $f: Fc \rightarrow d$, $g: Fc' \rightarrow d'$, $h: c \rightarrow c'$, $k: d \rightarrow d'$, in (1), the diagram on the left commutes if, and only if, the diagram on the right commutes:

$$\begin{array}{ccc}
 Fc & \xrightarrow{f} & d \\
 \downarrow Fh & & \downarrow k \\
 Fc' & \xrightarrow{g} & d'
 \end{array}
 \qquad
 \begin{array}{ccc}
 c & \xrightarrow{f'} & Gd \\
 \downarrow h & & \downarrow Gk \\
 c' & \xrightarrow{g'} & Gd'
 \end{array}
 \tag{1}$$

where, in the diagram on the right, $f' := \alpha_{c,d}(f)$ and $g' := \alpha_{c',d'}(g)$.

Exercise 37. A (bounded) *distributive lattice* is a poset L in which all finite suprema ($\vee$) and finite infima ($\wedge$) exist, and, for any $a, b, c \in L$, the equality $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ holds. A *morphism* from a distributive lattice L to a distributive lattice M is a function which preserves finite suprema and finite infima. This yields a category **DL**. We write **DL_{fin}** for the full subcategory on the finite distributive lattices.

- a. Prove that the forgetful functor $U: \mathbf{DL}_{\text{fin}} \rightarrow \mathbf{Set}_{\text{fin}}$ has a left adjoint.

Hint. For any finite set X , let FX be the lattice of upward closed subsets of the ‘cube’ poset $\{0, 1\}^X$, equipped with an appropriate partial order, and define $\eta_X: X \rightarrow UFX$ by sending $p \in X$ to $\eta_X(p) := \{v \in \{0, 1\}^X \mid v(p) = 1\}$.

Remark. This exercise implies that, for any $n \in \mathbb{N}$, the number $D(n) := |Fn|$ is finite. The number $D(n)$ is called the n^{th} *Dedekind number*. The exact value of $D(9)$ was only computed in 2023, and the exact value of $D(10)$ is not known.

- b. Does the extension $U: \mathbf{DL} \rightarrow \mathbf{Set}$ have a left adjoint?

11 Universal elements and arrows

In this section, let C be a category and $F: C^{\text{op}} \rightarrow \mathbf{Set}$ a functor.

Definition G. An *element* of the functor F is a pair (c, x) , where c is an object of C and x is an element of the set Fc . Given two elements (c, x) and (d, y) of the functor F , a *morphism* from (c, x) to (d, y) is a morphism $f: c \rightarrow d$ in C such that $Ff(y) = x$. We write $\int F$ for the category of elements of the functor F .

Exercise 38. Prove that $\int F$ is a category.

Definition H. An element (c, x) of F is *universal* if the unique (by the Yoneda Lemma) natural transformation, $\alpha: \mathbf{y}_c \Rightarrow F$, such that $\mathbf{e}_{c,F}(\alpha) = x$, is an isomorphism.

Exercise 39. Let (c, x) be an element of F . Show that (c, x) is universal if, and only if, (c, x) is a terminal object of $\int F$.

Exercise 40. This exercise shows how to turn around the arrows in the previous two exercises. Let D be a category and let $G: D \rightarrow \mathbf{Set}$ be a (covariant!) functor. Define $\int G$ to be the category whose objects are elements of G and whose morphisms $(c, x) \rightarrow (d, y)$ are the morphisms $f: c \rightarrow d$ in C such that $Gf(x) = y$. This category is also denoted $\int G$, although this notation can lead to confusion, see item a.

- a. Let $C := D^{\text{op}}$ and let $G': C^{\text{op}} \rightarrow \mathbf{Set}$ be the same functor as G , but viewed contravariantly. Show that the category $\int G$, as defined in this exercise, is the same as $(\int G')^{\text{op}}$, as defined in Definition G.
- b. Write $\mathbf{y}': D^{\text{op}} \rightarrow \mathbf{Set}^D$ for the Yoneda embedding of D^{op} and, for any object d of D , $\mathbf{e}'_{c,G}$ for the bijection $\mathbf{Set}^D(\mathbf{y}'c, G) \xrightarrow{\cong} Gc$. An element (c, x) of G is *universal* if the unique natural transformation $\alpha: \mathbf{y}'c \Rightarrow G$ with $\mathbf{e}'_{c,G}(\alpha) = x$ is an isomorphism. Prove that an element (c, x) of G is universal if, and only if, (c, x) is an initial object in $\int G$.

Conclusion: the meaning of ‘universal element’ depends on whether the functor is seen as covariant or contravariant.

Definition I. A *universal arrow* from an object c of C to a functor $U: D \rightarrow C$ is a pair (d, u) , where d is an object of D and $u: c \rightarrow Ud$ is a morphism such that, for any morphism $f: c \rightarrow Ue$, there exists a unique morphism $f': d \rightarrow e$ in D such that $Uf' \circ u = f$.

Exercise 41. Let $U: D \rightarrow C$ be a functor, let c be an object of C and consider the functor $H: D \rightarrow \mathbf{Set}$ given by composing U with $\mathbf{y}'c: C \rightarrow \mathbf{Set}$. Prove that a universal arrow is the same thing as a universal element of the covariant functor H .

12 Cartesian closed categories

Recall that a *cartesian closed category* is defined as a category C with finite products such that, for any object a of C , the functor $(-) \times a: C \rightarrow C$ has a right adjoint; we will write $(-)^a: C \rightarrow C$ for this right adjoint.³

Exercise 42. Let C be a category with finite products. Prove that C is cartesian closed if, and only if, for any objects a, b of C , there exist an object b^a in C and a morphism $\epsilon_{a,b}: b^a \times a \rightarrow b$ such that, for any morphism $f: c \times a \rightarrow b$, there exists a unique morphism $f': c \rightarrow b^a$ such that $\epsilon_{a,b} \circ (f' \times \text{id}_a) = f$.

Exercise 43. Let P be a partially ordered set. Characterize in order-theoretic terms what it means for the category associated to P to be cartesian closed.

Exercise 44 (Exam 2023). Let C be a cartesian closed category in which finite coproducts exist. Construct, for any objects a, b, c , a morphism $c^a \times c^b \rightarrow c^{a+b}$.

Exercise 45. Let P be a poset. An ω -chain in P is a countable sequence $(p_i)_{i \in \omega}$ such that $p_i \leq p_{i+1}$ for every $i \in \omega$. The poset P is ω -chain-complete if any ω -chain has a supremum in P . A function $f: P \rightarrow Q$ between ω -chain-complete posets is a *morphism* if it preserves suprema of ω -chains. Prove that the category of ω -chain-complete posets is cartesian closed.

Remark. One may use this cartesian closed category to obtain a model of the untyped λ -calculus, by constructing a non-trivial ω -chain-complete U that is isomorphic to both $U \times U$ and U^U . This method was pioneered by D. Scott (1972). For more information, see, e.g., Ch. 18 in J. Lambek, P. J. Scott, *Introduction to higher order categorical logic*, Cambridge University Press (1986).

Exercise 46. Let C be a small category. Recall that $\widehat{C}$ is the category of functors $C^{\text{op}} \rightarrow \mathbf{Set}$, with natural transformations between them. Prove that $\widehat{C}$ is cartesian closed.

Hint. First assume the exponential G^F exists, and use the Yoneda lemma to rewrite $G^F(a)$ as the set of natural transformations from $\mathbf{y}a$ to G^F .

³In the literature, you may also find the notations $[a \rightarrow b]$ or $a \Rightarrow b$ instead of b^a .